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A Branching Process Model for Sand Avalanches

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An analytically solvable model for sand avalanches of noninteracting grains of sand, based on the Chapman–Kolmogorov equations, is presented. For a single avalanche, distributions of lifetimes, sizes of overflows and avalanches, and correlation functions are calculated. Some of these are exponentials, some are power laws. Spatially homogeneous distributions of avalanches are also studied. Computer simulations of avalanches of interacting grains of sand are compared to the solutions to the Chapman–Kolmogorov equations. We find that within the range of parameters explored in the simulation, the approximation of noninteracting grains of sand is a good one.

KEY WORDS: Self-organized criticality; avalanches; branching processes; dynamical critical phenomena; $1/f$ noise.

1. INTRODUCTION

A signal whose power spectrum $S(f)$ scales over a broad range of frequencies as $1/f^{\beta}$, where β is of the order of unity, is called "$1/f$ noise." The great variety of examples of $1/f$ noise has made its explanation a challenge. Examples include earthquakes, the light from quasars, the intensity of sunspots, the current through resistors, the sand flow in an hourglass, the flow of the river Nile, interbeat intervals of the mammalian heart, stock exchange price indices, income distribution, scientific productivity, body weights, the performance of cooperative processes, and more.[1-7] In refs. 2 and 3 it is shown how "multiplicative processes" and "amplification processes" naturally produce a $1/f$ noise. This explanation suits the second half of the preceding examples well.

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The first half are systems through which something is flowing. This is known to lead to self-organization.[7] Bak *et al.*[1] have put forward sandslides as a paradigm for this class of systems. Based on numerical studies, they have argued that these systems have time correlation functions that decay according to a power law. This is one of the meanings that "self-organization" in time has, because time correlation functions usually decay exponentially, that is, faster than by a power law. It also implies $1/f$ noise, because if the correlation function behaves as $|t|^{-\alpha}$, its Fourier transform behaves as $|\omega|^{\alpha-1}$ (see Appendix A). Closed systems at the critical point are also known to have power-law-decaying correlation functions. However, a fine tuning is required to set them in the initial state, whereas the systems exemplified by sandslides evolve naturally toward it,[1,5,6] thus the word *self-organization.*

An interesting and simple discrete-time branching process model for avalanches has been presented by Alstrøm and its critical exponents have been derived.[8] In the present article, we discuss extensively a continuous-time branching process model. The theoretical advantage of the latter model is not only that in most situations time is better described by a continuous parameter, but also that in the former model synchronization between the "branching" of different grains of sand is assumed. Even though both models have the same $t \to \infty$ limit, as we shall show in Appendix B, this is not *a priori* obvious, since the particular discrete-time model presented in ref. 8 cannot be embedded in a continuous one (ref. 9, p. 102), that is, there is no continuous-time branching process which, considered at regular intervals of time, is equivalent to the discrete one.

In Section 2 we present the model. In Section 3 we show that if we make the approximation that the grains of sand are probabilistically independent, the model is equivalent to a Chapman–Kolmogorov equation that can be solved analytically. In Section 4 we calculate the distributions of lifetimes and sizes of overflows and avalanches. In Section 5 we find and discuss various time correlation functions. In Section 6 we consider an observer immersed in a random distribution of avalanches, in any number of dimensions. In Section 7 we explore numerically, the case of interacting grains of sand. We state our conclusions in Section 8.

2. THE MODEL

In our probabilistic model for avalanches, in an infinitesimal amount of time dt only three things can happen to a rolling grain of sand: (1) it might be absorbed, (2) it might extract another grain of sand, or (3) it might just roll on. Let the probabilities be $\mu(t)\,dt$, $\lambda(t)\,dt$, and $1-[\lambda(t)+\mu(t)]\,dt$, respectively. In the critical state, $\lambda(t)=\mu(t)$. How the critical state was arrived at is *not* part of this model. If the properties of the surface of the sandpile are everywhere the same, then μ and λ can only depend on the velocity of the rolling grain. Assume that at $t=0$ (the time at which the grain of sand is dropped), $\lambda(0)=\mu(0)$ and that the velocity of the grain of sand increases. Then absorption becomes less likely and extraction more likely. Thus,

$$\mu(t)<\mu(0)=\lambda(0)<\lambda(t)$$

Conversely, if the velocity were to diminish,

$$\mu(t)>\mu(0)=\lambda(0)>\lambda(t)$$

Thus, the velocity of the grain of sand is going to remain constant, and $\lambda=\mu$ is independent of time.

It is clear that the three above-mentioned possible infinitesimal fates of a grain of sand will, over a finite time, give rise to a "tree" whose branches will be the tracks left by the rolling grains of sand. Trees so defined are studied by the theory of branching processes.[9] For us this tree (or a constant time section of it) will be the avalanche. It can be embedded in any number of dimensions. Consequences of this embedding are discussed in Section 7. In the next section we obtain a complete description of the ensemble of these trees, provided different grains of sand are independent, by which we mean that each of them behaves as if the others did not exist. Again in Section 7 we explain why this is an approximation that works better in a large number of dimensions.

Finally, for the benefit of readers used to cellular automaton models of avalanches, one can think of the "states" of this model as being labeled only by j, the number of grains of sand rolling at time t, where t is the time since the initiating grain was dropped.

3. PROBABILISTIC INDEPENDENCE: THE CHAPMAN–KOLMOGOROV EQUATIONS

Under the assumption of the probabilistic independence of grains, which will be explained in the derivation that follows, the model has an analytical solution. Given μ, the probability $P_j(t)$ of finding j grains in motion at a time t after one grain of sand was dropped can be found. One way of doing this is by solving the backward Chapman–Kolmogorov equation, which we now review. (For a more complete discussion of the mathematics involved in this problem, see the book by Harris.[9] We warn the reader that the notation used in ref. 9 is similar to, but not the same as ours.)

Assume that the $P_j(t)$, $j \in N$, admit a Maclaurin expansion:

$$\begin{aligned} P_j(t) &= p_j t + o(t^2) \qquad (p_j \geq 0) \quad j \neq 1 \\ P_1(t) &= 1 + p_1 t + o(t^2) \qquad (p_1 < 0) \end{aligned} \tag{1}$$

For example, in our case we would have

$$p_0 = p_2 = \mu, \qquad p_1 = -2\mu \tag{2}$$

The normalization condition

$$\sum_{j=0}^{\infty} P_j(t) = 1, \qquad \forall t \in R^+ \tag{3}$$

implies

$$\sum_{j=0}^{\infty} p_j = 0 \tag{4}$$

We now define $P_{k \to j}(t)$ as the probability of finding j rolling grains at time $t' + t$ if there are k rolling at time t' or a time t after k grains of sand were dropped. We also define the generating function of the probabilities,

$$F(s, t) \equiv \sum_{j=0}^{\infty} P_j(t)\, s^j, \qquad s \in [0, 1] \tag{5}$$

Write now $P_j(t + dt)$ as follows:

$$P_j(t + dt) = \sum_{k=0}^{\infty} P_k(dt)\, P_{k \to j}(t) \tag{6}$$

$$P_j(t + dt) - P_j(t) = \sum_{k=0}^{\infty} p_k P_{k \to j}(t)\, dt \tag{7}$$

In general

$$P_{k \to j}(t) = \sum_{\substack{l_1, \dots, l_k \\ l_1 + \cdots + l_k = j}} P_{(l_1, \dots, l_k)}(t) \tag{8}$$

where $P_{(l_1, \dots, l_k)}(t)$ is the probability that, in a time t, the first grain of sand has produced l_1 grains of sand,..., and the kth grain, l_k grains of sand.

If the behavior of a grain of sand is not affected by the others, then $P_{(l_1, \dots, l_k)}(t)$ factorizes:

$$P_{(l_1, \dots, l_k)}(t) = P_{l_1}(t) \cdots P_{l_k}(t) \tag{9}$$

With this assumption

$$\frac{dP_j(t)}{dt} = \sum_{k=0}^{\infty} p_k \left(\sum_{\substack{l_1,\dots,l_k \\ l_1+\cdots+l_k=j}} P_{l_1}(t)\cdots P_{l_k}(t) \right) \tag{10}$$

$$\frac{\partial F(s,t)}{\partial t} = \sum_{k=0}^{\infty} p_k \sum_{j=0}^{\infty} \left(\sum_{\substack{l_1,\dots,l_k \\ l_1+\cdots+l_k=j}} P_{l_1}(t)\cdots P_{l_k}(t) \right) s^j$$

$$= \sum_{k=0}^{\infty} p_k [F(s,t)]^k \tag{11}$$

If we now define

$$h(s) \equiv \frac{\partial F(s,0)}{\partial t} \tag{12}$$

we can write Eq. (8) as

$$\frac{\partial F(s,t)}{\partial t} = h[F(s,t)] \tag{13}$$

which is the backward Chapman–Kolmogorov equation. In our case [from (2)]

$$h(s) = \mu(1-s)^2 \tag{14}$$

Since an appropriate choice of time units will make the numerical value of μ equal to one, we will set $\mu = 1$ in the theoretical discussions. In these units, the backward Chapman–Kolmogorov equation for our case reads

$$1 - 2F(s,t) + F(s,t)^2 = \frac{\partial F(s,t)}{\partial t} \tag{15}$$

with the initial condition

$$F(s,0) = s \tag{16}$$

which means that at time $t = 0$ we have one grain of sand. Equation (15) is of the Riccati type and can be solved by means of the substitution

$$F(s,t) = -\frac{\partial}{\partial t} \ln u(s,t) \tag{17}$$

The solution that satisfies the initial condition (16) is

$$F(s, t) = \frac{(1-s)\,t+s}{(1-s)\,t+1} \tag{18}$$

By differentiating Eq. (18), we find that

$$\begin{aligned} P_0(t) &= \frac{t}{1+t} \\ P_j(t) &= \frac{1}{t^2}\left(\frac{t}{1+t}\right)^{j+1}, \qquad j \neq 0 \end{aligned} \tag{19}$$

The expected number of grains of sand is

$$\sum_{j=0}^{\infty} jP_j(t) = \frac{\partial F(1, t)}{\partial s} \tag{20}$$

which is equal to one at all times, consistent with the defining condition for the critical state, $\lambda(t) = \mu(t)$. However, $\lim_{t \to \infty} P_0(t) = 1$, which means that, with probability one, any avalanche will die.

4. DISTRIBUTIONS OF LIFETIMES AND SIZES OF OVERFLOWS AND AVALANCHES

The probability density of lifetimes of the avalanches is

$$\rho(t) = \frac{dP_0(t)}{dt} = \frac{1}{(1+t)^2} \xrightarrow[t \to \infty]{} \frac{1}{t^2} \tag{21}$$

Thus this analytic model predicts a power law behavior for the lifetimes, with an exponent of -2. The computer simulations done in ref. 1 yield exponents of -0.43 and -0.92 for dimensions of 2 and 3. But these exponents cannot be asymptotic because then the corresponding tails would subtend an infinite area. Thus our model and the original one of Bak, Tang, and Wiesenfeld may still be asymptotically equivalent.

If we assume that time is proportional to the number of generations, the discrete-time model also yields a t^{-2} distribution for the lifetimes.[8]

Consider now a conical sandpile with critical slope sitting on a disk whose diameter is equal to the diameter of the base of the pile. We are interested in the distribution of sizes of the overflows that result when a grain of sand is added at the top. (The size of the overflow has been called the "drop number" D in other models.[10]) A useful picture is to see each avalanche as a tree rooted at the top of the sandpile. Then D is the number of branches that reach the edge of the sandpile. If we assume, as implied by Section 2, that the component along the generatrix of the velocity of the grains of sand as they roll down is constant and choose units of length such that its numerical value is 1, then in Eq. (19),

$$P_j(t) = \frac{1}{t^2}\left(\frac{t}{1+t}\right)^{j+1}, \qquad j \neq 0$$

t is the length of the generatrix of the cone and an exponential distribution for D is predicted. The same is true for any discrete-time model (ref. 9, p. 22) (not only for the one presented in ref. 8). Experiments performed so far with real sand remain inconclusive.[11,12]

We now turn our attention to the sizes of the avalanches. The definition of "size" of an avalanche depends on the model used. (For cellular automaton models it is often referred to as "the flipping number" F and it is defined in ref. 10.) In our case, think of the paths traced by the grains of sand as branches of a tree, all of which have the same cross section. The volume of that tree would then be the size of the avalanche.

One way to compute it would be as follows. Cut the tree at some regular spacing. Consider the spacing and the total number of branches cut. If we let the spacing go to zero, then the product of these two quantities will tend to the volume of the tree, except for a constant which is unimportant for our purposes.

We shall now calculate along the lines of the preceding paragraph (see ref. 9 and references therein). First, the generating function for the probability of cutting j_1 branches at time Δt and j_2 branches at time $2\,\Delta t$ is

$$\begin{aligned}\sum_{j_1, j_2 = 0}^{\infty} & P_{j_1}(\Delta t)\, P_{j_1 \to j_2}(\Delta t)\, s_1^{j_1} s_2^{j_2} \\ &= \sum_{j_1=0}^{\infty} P_{j_1}(\Delta t)\, s_1^{j_1} \sum_{j_2=0}^{\infty} P_{j_1 \to j_2}(\Delta t)\, s_2^{j_2} \\ &= F(s_1 F(s_2, \Delta t), \Delta t) \end{aligned} \tag{22}$$

By induction, the generating function for the probability of finding j_1 branches at time Δt, j_2 branches at time $2\,\Delta t$,..., j_N branches at time $N\,\Delta t$ is

$$F(s_1 F(\dots s_{N-1} F(s_N, \Delta t), \Delta t), \dots, \Delta t) \tag{23}$$

Let now $Q(N\,\Delta t, s)$ be the generating function for the probability of the sum $j_1 + \cdots + j_N$ to be equal to m. If

$$Q(s, N\,\Delta t) \equiv \sum_{m=0}^{\infty} Q_m(N\,\Delta t)\, s^m \tag{24}$$

then

$$Q_m(N\,\Delta t) = \sum_{j_1 + \cdots + j_N = m} P_{j_1}(\Delta t)\, P_{j_1 \to j_2}(\Delta t) \cdots P_{j_{N-1} \to j_N}(\Delta t) \qquad (25)$$

and

$$Q(s, N\,\Delta t) = F(sF(sF(\ldots sF(s, \Delta t), \Delta t), \ldots, \Delta t)) \qquad (26)$$

where F appears N times. The limit

$$Q(s) \equiv \lim_{N \to \infty} Q(s, N\,\Delta t) \qquad (27)$$

exists (this can be heuristically understood from the fact that the probability of extinction for a critical branching process is one) and, from (26), satisfies the functional equation

$$Q(s) = F(sQ(s), \Delta t) \qquad (28)$$

which can be easily solved for our case, as well as for the discrete-time case considered in ref. 8.

However, for the purposes of establishing that the sizes of the avalanches are distributed according to a power law, it suffices to quote a theorem by Otter,[13,9] which states that, for critical processes, the solution to (28) satisfies

$$Q_m \sim m^{-3/2} \qquad \text{as} \quad m \to \infty \qquad (29)$$

This result also applies to discrete-time branching processes, since $F(s, \Delta t)$ can obviously be considered as the generating function for such a process. The relation between (29) and critical exponents is discussed in ref. 8.

5. CORRELATION FUNCTIONS

Very often, in systems which might be modeled by some sort of chain reaction process, only $P_j(t)$ is observable, for some fixed t. Examples include light from quasars and the intensity of sunspots. However, if not only its final outcome, but also the avalanche itself is observable, correlation functions can be measured. In this section we will calculate correlation functions for a single avalanche and for a succession of avalanches at regular intervals.

Since the avalanches have, with probability one, a finite lifetime, the process at hand is neither ergodic nor stationary (in the sense of ref. 14). Therefore, the correlation functions will be calculated by performing an ensemble average, and we will not average over time.

The ensemble average value of the product of the number of grains at time t', $\#(t')$, times the number of grains at times $t'+t$, $\#(t'+t)$, is

$$\begin{aligned}
&\langle \#(t')\ \#(t+t') \rangle \\
&= \sum_{j=0}^{\infty} jP_j(t') \sum_{k=0}^{\infty} kP_{j\to k}(t) \\
&= \sum_{j=0}^{\infty} jP_j(t') \frac{\partial F^j}{\partial s}(1, t) = \sum_{j=0}^{\infty} j^2 P_j(t')\, F^{j-1}(1, t) \frac{\partial F}{\partial s}(1, t) \\
&= \sum_{j=0}^{\infty} j(j-1)\, P_j(t') + \sum_{j=0}^{\infty} P_j(t') = 1 + \frac{\partial^2 F}{\partial s^2}(1, t') \\
&= 1 + 2t'
\end{aligned} \tag{30}$$

This correlation function does not depend on t and diverges as $t' \to \infty$. Though at first sight these facts may be surprising, they are not particular to this model and are actually quite general.

The correlation function of any critical branching process does not depend on t. To see this, simply take the derivative

$$\begin{aligned}
&\frac{\partial}{\partial t} \langle \#(t')\ \#(t'+t) \rangle \\
&\qquad = \sum_{j=0}^{\infty} jP_j(t') \frac{\partial}{\partial t} \left(\sum_{k=0}^{\infty} kP_{j\to k}(t) \right) = 0
\end{aligned} \tag{31}$$

since the term in parentheses is the expected value for j avalanches, which is j. [We have evaluated Eq. (30) in a slightly longer way to illustrate techniques to be mentioned later.]

The second property of the correlation function (30) holds for any nontrivial [i.e., $P_1(t) \neq 1$] critical process. For such processes the extinction probability is 1,[9] or, in other words,

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} P_j(t) = 0 \tag{32}$$

which implies

$$\begin{aligned}
&\lim_{t\to\infty} \sum_{j=0}^{\infty} j^2 P_j(t) = \lim_{t\to\infty} \sum_{j=m}^{\infty} j^2 P_j(t) \\
&\qquad > m \lim_{t\to\infty} \sum_{j=m}^{\infty} jP_j(t) = m, \qquad \forall m \in N
\end{aligned} \tag{33}$$

The criticality of the process, $\sum_{j=0}^{\infty} jP_j(t) = 1$, and (30) imply

$$\lim_{t \to \infty} \sum_{j=m}^{\infty} jP_j(t) = 1, \qquad \forall m \in N \tag{34}$$

We can now apply these results to the correlation function:

$$\begin{aligned} &\lim_{t' \to \infty} \langle \#(t') \, \#(t+t') \rangle \\ &= \lim_{t' \to \infty} \sum_{j=0}^{\infty} jP_j(t') \sum_{k=0}^{\infty} kP_{j \to k}(t) \\ &= \lim_{t' \to \infty} \sum_{j=0}^{\infty} j^2 P_j(t') = \lim_{t' \to \infty} \sum_{j=m}^{\infty} j^2 P_j(t') > m, \qquad \forall m \in N \end{aligned} \tag{35}$$

which means that the limit is infinite, since m is arbitrary. Note that the process does not have to be critical for the preceding proof to go through. It is sufficient that $\lim_{t \to \infty} \langle \#(t) \rangle \neq 0$.

The intuitive idea behind the proof is that if $\lim_{t \to \infty} P_0(t) = 1$ but $\lim_{t \to \infty} \langle \#(t) \rangle \neq 0$, then, for large times, the probability $1 - P_0(t)$ of nonextinction has "migrated" to very large j's. If instead of computing the first moment of $\{P_j(t)\}_{j \in N}$, we compute the second moment, which is essentially what the correlation function is, then the result must be infinite.

Finally we want to mention that the knowledge of $\langle \#(t') \, \#(t'+t) \rangle = 1 + 2t''$ does not give much information about the possible values of $\#(t') \, \#(t+t')$, since their dispersion is very large. A calculation based on the techniques of Eq. (30) shows that

$$\begin{aligned} &\langle [\#(t') \, \#(t'+t) - \langle \#(t') \, \#(t'+t) \rangle]^2 \rangle \\ &= 24t'^3 + 12t'^2 t + 32t'^2 + 12tt' + 10t' + 2t \end{aligned} \tag{36}$$

In practice, however, one does not observe avalanches of unbounded lifetime. It is therefore interesting to calculate correlation functions for avalanches of fixed lifetime T. If we introduce the variable $t'' = T - t - t'$, we have

$$\begin{aligned} &\langle \#(t') \, \#(t'+t) \rangle_{\text{lifetime} = T} \\ &= \frac{\sum_{j=0}^{\infty} jP_j(t') \sum_{k=0}^{\infty} kP_{j \to k}(t) [d(P_0)^k/dt''](t'')}{\sum_{j=0}^{\infty} P_j(t') \sum_{k=0}^{\infty} P_{j \to k}(t) [d(P_0)^k/dt''](t'')} \end{aligned} \tag{37}$$

because $[d(P_0)^k/dt''](t'')$ is the probability density of death for an avalanche that had k grains at time $t'' = 0$. Expression (37) can be evaluated using techniques similar to the ones used in the evaluation of expression (30). We obtain

$$
\begin{aligned}
&\langle \#(t')\ \#(t+t')\rangle_{\text{lifetime}=t+t'+t''} \\
&\quad = \left(\frac{\partial}{\partial t''} \left\{ P_0(t'') \frac{\partial F}{\partial s}(t, P_0(t'')) \right.\right. \\
&\qquad \left.\left. \times \left[F(t, P_0(t'')) \frac{\partial^2 F}{\partial s^2}(t', F(t, P_0(t''))) + \frac{\partial F}{\partial s}(t', F(t, P_0(t''))) \right] \right\} \right) \\
&\qquad \times \left[\frac{\partial}{\partial t''} F(F(P_0(t''), t), t') \right]^{-1} \\
&\quad = \frac{(1+t)^2 + 2t(2+2t't''+t)(t'+t'') + (1+2t)(t'^2+t''^2) + 4t(t+2)\,t't''}{(1+t'+t+t'')^2} \\
&\qquad + \frac{(6t't''+2)(t'+t'') + 2t't''(3t't''+4)}{(1+t'+t+t'')^2}
\end{aligned} \tag{38}
$$

Notice that $\langle \#(t')\ \#(t'+t)\rangle_{\text{lifetime}=t'+t+t''}$ is symmetric in t' and t''. This was to be expected since the ensemble of the functions $\#(t)$ is invariant under time reversal. This is so because time reversal switches the absorption and extraction probabilities, which at the critical state are the same. Of course, this symmetry also holds for the discrete-time model presented in ref. 8.

It is shown in ref. 1 that if the correlation function of an avalanche of lifetime T is $\sim e^{-t/T}$, then a power law decay in the probability density of lifetimes would yield $1/f$ noise (see Appendix A). $\langle \#(t')\ \#(t'+t)\rangle_{\text{lifetime}=t'+t+t''}$ is not an exponential, but it still yields a nondecaying power law (for $t' \gg 1$) when weighted by $d(P_0)(t'')/dt'' = 1/(1+t'')^2$ and integrated over t'':

$$
\begin{aligned}
&\langle \#(t')\ \#(t'+t)\rangle \\
&\quad = 1 + 2t' \\
&\quad = \int_0^\infty dt'' \frac{dP_0(t'')}{dt''} \langle \#(t')\ \#(t'+t)\rangle_{\text{lifetime}=t'+t+t''}
\end{aligned} \tag{39}
$$

The correlation function for a sandpile at the critical state under a constant flux of grains of sand has been calculated elsewhere.[15]

6. RANDOM DISTRIBUTION OF AVALANCHES

The purpose of this section is to illustrate how in certain geometrical scenarios the grains of sand that are collected come from avalanches of different ages. This could provide an explanation for $1/f$ noise *à la* Van der Ziel [see expression (A.5) and ref. 16] if the exponentials in j from Eq. (19) are integrated with an appropriate time-dependent kernel. We have not found such an explanation, but other natural scenarios might provide one.

We present two such cases. In the first one our approximation of independence does not apply well, because the grains of sand are going to be rolling along a straight line and thus interacting. It is, however, completely solvable.

First, consider a cylindrical box with a hole in the center of its bottom. There is an inverted sandpile in it, which has the critical slope. Grains of sand are added at random locations. The only difference between this and the top half of an hourglass whose lid has been reoved and on which sand is raining is that the bottom walls of our box are not tilted, so that not all of the sand will flow out. See Fig. 1.

Let T be the radius of our box in a system of units in which the average horizontal velocity of a grain in a sandslide is equal to unity. Assume that grains of sand belonging to different avalanches are collected during nonoverlapping intervals of time. This will be the case if the rate at

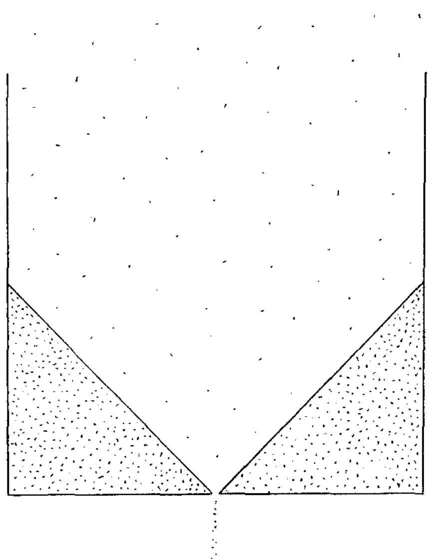


Fig. 1. One grain of sand is added at a random location to the sand contained in a cylindrical box with a hole in the center of its bottom. The resulting overflow which falls through the hole is collected and counted.

which new grains of sand are added and the dispersion in velocities of rolling grains of sand are sufficiently small and if the average velocity is sufficiently large. Then, denoting by $\Pi'(j, T)$ the probability for an overflow to consist of j grains,

$$\Pi'(j, T) = \frac{1}{\pi T^2} \int_0^T dt \, 2\pi t P_j(t) \tag{40}$$

Since $\sum_{j=0}^{\infty} j^k P_j(t) = 1$ for $k = 0, 1$, the $\Pi'(j, T)$ are also normalized and their expectation value is also one. From Eq. (17) it is easy to check that

$$\Pi'(0, T) = 1 - \frac{2}{T} + \frac{2}{T^2} \ln(1 + T) \tag{41}$$

and

$$\Pi'(j, T) = \frac{2}{T^2} \sum_{l=j+1}^{\infty} \frac{\xi^l}{l}, \qquad j \in N^* = \{1, 2, 3,...\} \tag{42}$$

where

$$\xi \equiv \frac{T}{1+T} \tag{43}$$

Upper and lower bounds for the functions $\Pi'(j, T)$, $j = 1, 2, 3,...$, in terms of the exponential integral function $\mathrm{Ei}(x)$ (ref. 17, p. 927) have been given elsewhere.[15]

In Section III.C of ref. 1 a similar situation (the difference being that in ref. 1 the sand overflows at the edges of the circle, instead of at the center) is numerically simulated (the simulation is based on an approach to the problem different than ours) and their result is that $\Pi'(j, T) \sim j^{-\alpha}$, where $\alpha \simeq 1$. Plots of $\ln \Pi'(j, T)$ against $\ln j$ (see Fig. 2) show that for large j and T, $\Pi'(j, T)$ can be approximated by $j^{-\alpha(T)}$. However, $\alpha(T)$ is in fact a decreasing function of j and can be very different from 1.

There is a recurrence relation for the $\Pi'(j, T)$ for higher dimensions. If we denote by $\Pi'_n(j, T)$ the probability for collecting j grains in a situation analogous to the one previously described, but in n dimensions,

$$\Pi'_n(j, T) = \frac{n}{T^n} \int_0^T dt \, t^{n-1} P_j(t) \tag{44}$$

then the following relations hold[15]:

$$\Pi'_n(0, T) = 1 - \frac{n}{(n-1)\,T} \Pi'_{n-1}(0, T) \tag{45}$$

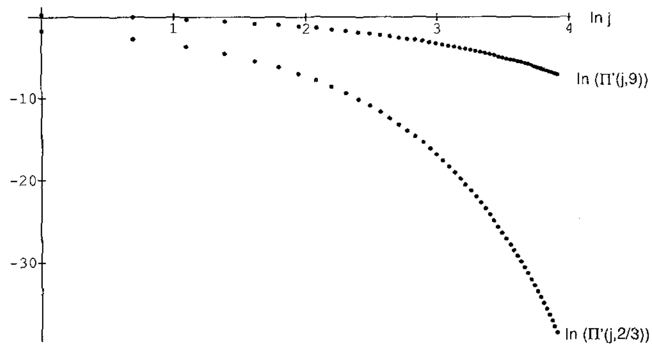


Fig. 2. Logarithmic plots of $\Pi'(j, 9)$ and $\Pi'(j, 2/3)$ for $j = 1, 2, \ldots, 40$. $\Pi'(j, 9)$ is the upper curve. Notice that the greater j and T are, the wider is the range over which $\Pi'(j, T)$ can be approximated by a power law. If $\Pi'(j, T) \sim j^{-\alpha}$, $\alpha \simeq -5$ for $(j, T) = (40, 9)$ and $\alpha \simeq -40$ for $(j, T) = (40, 2/3)$.

and

$$\Pi'_n(j, T) = \frac{n}{n-2} \frac{1}{T^2} \frac{T^j}{(1+T)^j} - \frac{n(n+j-2)}{(n-1)(n-2)\,T} \Pi'_{n-1}(j, T) \tag{46}$$

Now consider the other case, where the grains of the avalanches, instead of going to the hole in the center, propagate isotropically (we can image that in the previous experiment the walls have been removed). We are still interested in the overflow collected at the center. This particular model is not realizable with sand. We are, however, not interested in sand avalanches *per se*, but as paradigms of self-organized criticality.[1,5,6] Furthermore, this model is natural if one thinks of a chain reaction (e.g., in a piece of fissionable material).

If we denote by $\Pi_n(j, T)$ the function that now plays the role of the $\Pi'_n(j, T)$, $\Pi_n(j, T)$ will again be calculated by means of an integral between 0 and T. We now explain that the contribution of spherical shells of radius t to $\Pi_n(j, T)$, for large j's and a not too small t is proportional to

$$t^{(n-1)} t^{(n-1)} P_{j[2\Pi^{n/2}/\Gamma(n/2)]\, t^{(n-1)}}(t^2) \tag{47}$$

The factor t^{n-1} is proportional to its area. The other factor t^{n-1} comes from the normalization condition:

$$\sum_{j=0}^{\infty} \Pi_n(j, T) = \int_0^T dt\, \frac{2\pi^{n/2}}{\Gamma(n/2)}\, t^{n-1} \tag{48}$$

The argument is t^2 because it takes the avalanche a time proportional to t^2 to reach the center if we assume that the grains take on a random direction after each collision. The subscript of P, $[2\Pi^{n/2}/\Gamma(n/2)]\, t^{n-1}$, is the area of an $(n-1)$-dimensional spherical shell. It takes into account how the density of grains thins out as the avalanche progresses.

If j were not large, then avalanches whose total number of grains of sand is different from $j[2\Pi^{n/2}/\Gamma(n/2)]\, t^{n-1}$ could give, by chance, nonzero contributions. If t is small, then $j[2\Pi^{(n/2)}/\Gamma(n/2)]\, t^{n-1}$ can become <1, and then the functional form of the probability changes. Also, the finite size of the collecting hole would play a role.

It follows from (19) and (47) that the contribution of distant shells to $\Pi_n(j, T)$ for large j's is going to be very small. Thus, as we increase T, the change in size of the system will affect only the $\Pi_n(j, T)$ of smaller and smaller j's. The $\Pi_n(j, T)$ of high j's become saturated until, beyond a certain T, only $\Pi_n(1, T)$ changes with size.

7. THE MODEL WITHOUT THE PROBABILISTIC INDEPENDENCE APPROXIMATION

In Section 2 we assumed the probabilistic independence of grains of sand. Due to this approximation the Chapman–Kolmogorov equations describe in fact an avalanche taking place in an infinite number of dimensions, where the probability for any two grain paths to meet is zero. One can ask if there exists a dimension above which the infinite-time limit of a "real" avalanche would be correctly described by the Chapman–Kolmogorov equations. This question will be addressed elsewhere.[18]

To go beyond the probabilistic independence approximation we have studied avalanches numerically. Before we present our results we must justify that the simulation of a discrete-time branching process can also be used to study continuous-time branching processes. We have done so in Appendix B.

The avalanches have been simulated as follows. In a rectangle, absorption and extraction sites have been randomly distributed with a density of μ each. A grain of sand starts at the midpoint of the top side of the rectangle and moves down with a probability p and to either side with a probability $(1-p)/2$. We have done simulations for the following pairs of values of (μ, p): $(0.01, 0.9)$, $(0.05, 0.9)$, and $(0.01, 0.5)$. The length of the rectangle has always been 471 sites, and its width has been 50 sites for $p = 0.9$ and 100 sites for $p = 0.5$. For each pair we have done simulations under two different conditions: regeneration and nonregeneration of the absorption/extraction sites. The idea of regeneration is that if the absorption and extraction sites continue to exist even after a grain of sand has visited them, then tracks of different grains cannot cross, because they do not leave tracks in the first place. We know that this trick does not eliminate all the correlations due to the finite dimensionality, because it is still true that grains that are near each other are going to visit similar configurations of absorption/extraction sites. Nonregeneration of absorption/extraction sites corresponds, of course, to the simulation of a two-dimensional avalanche.

We believe that in a finite number of dimensions the lifetimes of the avalanches are going to be longer, for the following reason: A grain of sand that is on the path that was taken by another grain is not going to be absorbed or is not going to extract another grain, because if either were possible the path would have ended or bifurcated, respectively. This situation creates some waiting time for the grain. Our own computer simulations in two dimensions have yielded an exponent of -2 for the distribution of lifetimes (Fig. 3), as predicted by equation (21). We do not know if this means that the above-mentioned effect is still too small to be detected, or if it means that the asymptotic exponent is going to be unaffected.

Probably the most direct way to test the validity of the approximation is to compare the expressions (17) for $P_j(t)$, $j = 0, 1, 2, \ldots$, to the ones obtained from the simulation (see Fig. 4). We have done so for all six (μ, p)

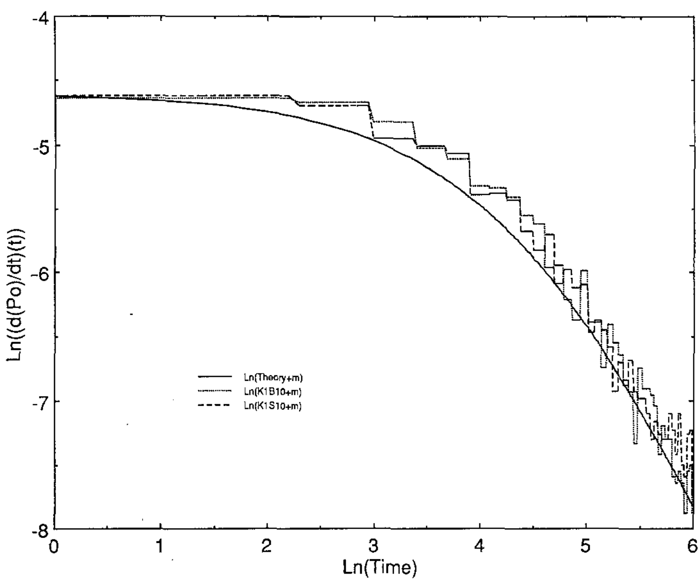
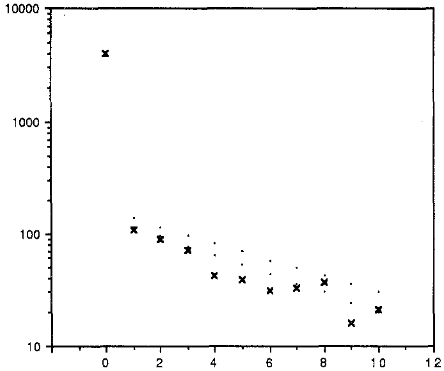
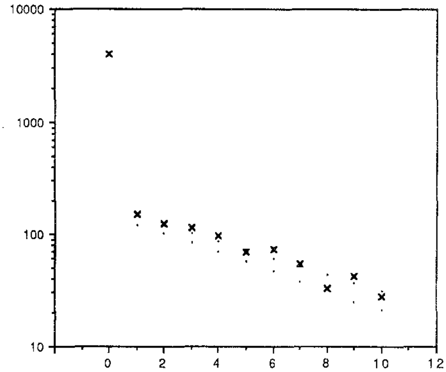


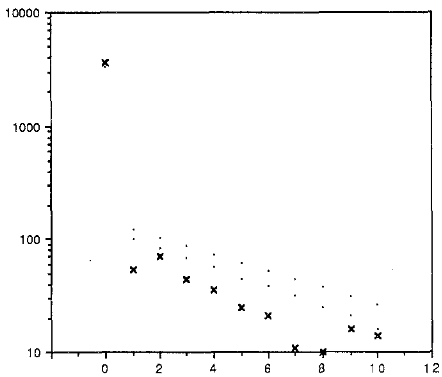
Fig. 3. Logarithmic plot of the lifetime probability density. The continuous curve is the analytical prediction; the dotted and dashed lines have been obtained from simulations with and without regeneration, respectively. The parameters are $(\mu, p) = (0.01, 0.9)$, and 4000 simulations of lifetimes $\leqslant 628$ have been used for each of the experimental curves.



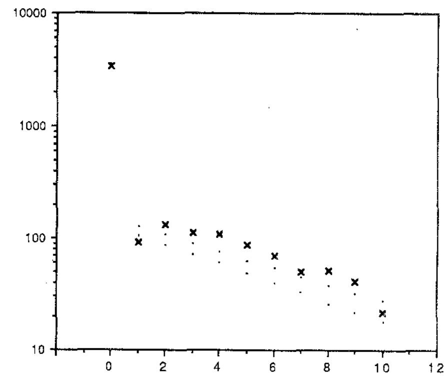
(a) (μ,p) = (0.01, 0.9), t = 500, regeneration



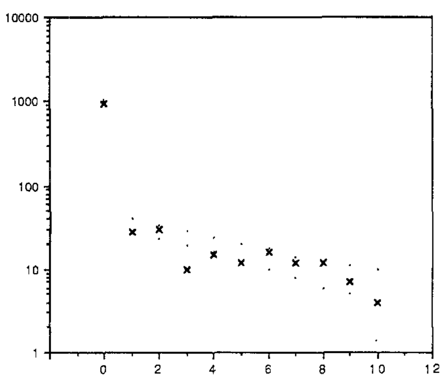
(b) (μ, p) = (0.01, 0.9), t = 500, no regeneration



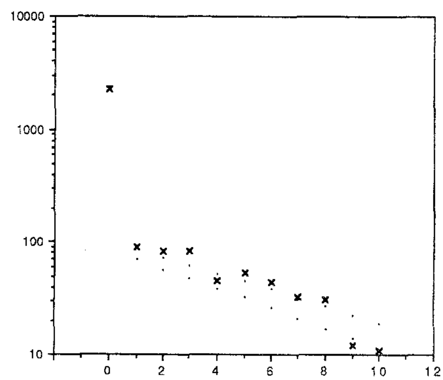
(c) (μ,p) = (0.05, 0.9), t = 100, regeneration



(d) (μ,p) = (0.05, 0.9), t = 100, no regeneration



(e) (μ, p) = (0.01, 0.5), t = 500, regeneration



(f) (μ, p) = (0.01, 0.5), t = 500, no regeneration

Fig. 4. The number of avalanches with a given number of grains of sand, plotted on a logarithmic scale, for six different conditions. The cross is the result of the simulation, while the two dots are the values given by the Chapman–Kolmogorov equations $\pm\sigma$. Sometimes these symbols are on the top of another and not all three of them are distinguishable. The logarithmic scale is convenient because of the large difference between the first value and the rest, and because the Chapman–Kolmogorov equations predict a geometric decrease for the values different from the first. However, it can be misleading in estimating the magnitude of the errors. In particular, the deviations for the first value are much larger than they seem. For (a)–(f) they are approximately $+6$, -1.5, $+13$, -5, -3 and -3 σ's. Note that the parameter $\mu t = 5$ for all six figures; (no) regeneration refers to the absorption/extraction sites.

pairs for various times. There is an overall good agreement. The deviations were of the order of 0 to 2 σ's for $\mu = 0.01$ and 0 to 5 σ's for $\mu = 0.05$ [except for the $P_0(t)$, for which they were about twice as large as for the $P_j(t)$, $j \neq 0$].

A more intensive numerical study is needed to study the statistically significant deviations from the Chapman–Kolmogorov predictions. Note that since for a discrete-time branching process the generating function at time step n is equal to the nth iterate of the generating function at time step 1,[9] the present section could be read as a study of whether our computer simulation would be a good Monte Carlo method for iterating polynomials of positive coefficients.

8. CONCLUSIONS

For a single avalanche, branching process models based on the Chapman–Kolmogorov equations or on its discrete-time counterpart, called the Galton–Watson process,[9] predict a t^{-2} distribution of lifetimes, an exponential decay in the size of the overflows, and a power law distribution, with an exponent of $-3/2$, for the sizes of the avalanches.

The time correlation functions found are polynomials or rational functions which display power law behavior for large times. They lack a time scale, however, in a more trivial and fundamental way, which comes from the fact that the variables μ and t always appear as the product μt, since this is the way in which they appear in the infinitesimal process. This means that if μ is unknown, a continuous-time critical branching process cannot be used to calibrate a clock. Obviously, this symmetry is not shared by the discrete-time model.

The computer simulations suggest that the approximation of probabilistic independence in not a crude one, and that, for a certain range of parameters, the Chapman–Kolmogorov equations could describe real phenomena.

APPENDIX A

Let a correlation function be of the form

$$|t|^{-\alpha}, \qquad t \in R, \quad \alpha \in R \tag{A.1}$$

Of course, if the signal is of finite amplitude, expression (A.1) can only be an approximation to the real correlation function.

The power spectrum corresponding to $|t|^{-\alpha}$ is

$$S(\omega)=\int_{-\infty}^{+\infty} dt\,|t|^{-\alpha}\,e^{-i2\pi\omega t}$$

$$=k^{\alpha-1}\int_{-\infty}^{+\infty} d(kt)\,|kt|^{-\alpha}\,e^{i2\pi(\omega/k)kt}=k^{\alpha-1}S\left(\frac{\omega}{k}\right) \tag{A.2}$$

where k is a real positive number. Thus the power spectrum $S(\omega)$ will be the solution to the functional equation

$$S(\omega)=k^{\alpha-1}S\left(\frac{\omega}{k}\right) \tag{A.3}$$

whose unique solution is (ref. 20, p. 319)

$$S(\omega)=S(1)\,\omega^{\alpha-1} \tag{A.4}$$

One instance in which the correlation function is of the form $|t|^{-\alpha}$ is when there are exponentially decaying processes with lifetimes distributed according to a power law. This can be seen from the following integral:

$$|t|^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^{\infty} d\tau\,\tau^{-1-\alpha}e^{-|t|/\tau} \tag{A.5}$$

APPENDIX B

We will show here that the expressions for $P_j(t)$ for the discrete-time model have the same $t\to\infty$ limit as (19). Now rewrite (19), together with its $t\to\infty$ limits, in a fashion more appropriate to follow the argument of the Appendix. Also, the density μ will be explicitly written. We have

$$P_0(t)=\frac{\mu t}{1+\mu t}\xrightarrow[t\to\infty]{} 1-\frac{1}{\mu t}+O\left(\left(\frac{1}{\mu t}\right)^2\right)$$

$$P_j(t)=\left(\frac{1}{\mu^2t^2}\left(\frac{\mu t}{1+\mu t}\right)\right)\left(\frac{\mu t}{1+\mu t}\right)^j,\qquad j\neq 0$$

$$\lim_{t\to\infty}\frac{1}{\mu^2t^2}\left(\frac{\mu t}{1+\mu t}\right)=\frac{1}{\mu^2t^2}+O\left(\left(\frac{1}{\mu t}\right)^3\right) \tag{B.1}$$

Let us denote by $P_j(n)$, $n=1, 2, 3,...,$ the corresponding discrete-time expressions. $P_0(n)$ satisfies[9]

$$P_0((n+1))=f(P_0(n)) \tag{B.2}$$

where

$$f(s) \equiv \mu s^2 + (1 - 2\mu)\, s + \mu \tag{B.3}$$

is called the generating function and μ is the probability of death or splitting in a step of time. Equation (B.2) can be written

$$\frac{P_0((n+1)) - P_0(n)}{(n+1) - n} = \mu(P_0(n) - 1)^2 \tag{B.4}$$

If we now pretend that n is a continuous variable, Eq. (B.4) becomes a differential equation whose solution for the boundary conditions $P_0(1) = \mu$, $P_0(\infty) = 1$ [since for the process specified by Eq. (B.3) the probability of extinction is 1; see ref. 9, p. 7] satisfies

$$\lim_{n \to \infty} P_0(n) = 1 - \frac{1}{\mu n} \tag{B.5}$$

like the corresponding term of Eq. (B.1).

The $n \to \infty$ form of $P_j(n)$ for $j \neq 0$ can be obtained from a theorem due to Yaglom[19] (see ref. 9, p. 22). It is

$$\begin{aligned}\lim_{n \to \infty} P_j(n) &= \lim_{n \to \infty} \sum_{l=j}^{\infty} P_l(n) - \lim_{n \to \infty} \sum_{l=j+1}^{\infty} P_l(n)\\ &= \lim_{n \to \infty} [1 - P_0(n)]\, e^{-(j+1)/n\mu} - \lim_{n \to \infty} [1 - P_0(n)]\, e^{-j/n\mu}\\ &= \frac{1 - e^{-1/n\mu}}{n\mu} (e^{-1/n\mu})^j\end{aligned} \tag{B.6}$$

where in the last equality Eq. (B.5) has been used. Here

$$\lim_{n \to \infty} \frac{1 - e^{-1/n\mu}}{n\mu} = \frac{1}{\mu^2 n^2} + O\left(\left(\frac{1}{\mu n}\right)^3\right)$$

$$\lim_{n \to \infty} e^{-1/n\mu} = 1 - \frac{1}{n\mu} + O\left(\left(\frac{1}{\mu n}\right)^2\right)$$

like the corresponding parts of Eq. (B.1).

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